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NEW FORM OF THE EQUATIONS OF MOTION OF A VISCOUS FLUID IN LAGRANGIAN VARIABLES

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An equation is derived which describes the motion of an incompressible viscous fluid in Lagrangian variables. Velocity circulations over contours enclosing infinitesimal areas in each of the planes of the Lagrangian variables (Cauchy invariants) are taken to be unknown functions. Examples of exact solutions obtained using this method of flow description are analyzed.

Key words: *viscosity, Cauchy invariants, vorticity, Lagrangian variables.*

Introduction. A derivation of the Navier–Stokes equations in Lagrangian variables is given in [1, 2]. The continuity equation for homogeneous incompressible fluid flows is written as

$$[X_1, X_2, X_3] = D_0(a, b, c). \tag{1}$$

Here $X_1 = X(a, b, c, t)$, $X_2 = Y(a, b, c, t)$, and $X_3 = Z(a, b, c, t)$ are the coordinates of fluid-particle trajectories and a, b, and c are Lagrangian variables. The square brackets denote the Jacobian in the Lagrangian variables. The function D_0 is a function of the Lagrangian variables and does not depend on time t. If the initial position of a fluid particle coincides with the Lagrangian coordinates, i.e., $X_0 = a$, $Y_0 = b$, and $Z_0 = c$, then $D_0 = 1$.

The equations of motion are written as

$$\frac{\partial^2 X_i}{\partial t^2} = -(\rho D_0)^{-1}[X_j, X_k, p] + \nu D_0^{-1}\Big\{\Big[X_2, X_3, D_0^{-1}\Big[X_2, X_3, \frac{\partial X_i}{\partial t}\Big]\Big]$$
$$+ \Big[X_3, X_1, D_0^{-1}\Big[X_3, X_1, \frac{\partial X_i}{\partial t}\Big]\Big] + \Big[X_1, X_2, D_0^{-1}\Big[X_1, X_2, \frac{\partial X_i}{\partial t}\Big]\Big]\Big\}. \tag{2}$$

Here ρ is the density, p is the pressure, and ν is the kinematic viscosity. Equations (1) and (2) for the unknowns $\boldsymbol{X}(\boldsymbol{a}, t) = X_i(a, b, c, t)$, where $i = 1$, 2, and 3, and $p(a, b, c, t)$ constitute the complete system of dynamic equations for an incompressible viscous fluid in the Lagrangian variables. A derivation of these equations was first given in [3]. Later, the same result was obtained independently by Monin [4] (see also [1]). In [1, 3, 4], it was assumed that $D_0 = 1$.

In the present paper, we propose a new method of representing the equation of motion (2) in which the unknown functions are the velocity circulations over the contours enclosing infinitesimal areas in each of the planes of the Lagrangian variables. Examples of exact solutions obtained using the new method of representing the equations of viscous fluid dynamics are considered.

1. Equations for the Cauchy Invariants. The equations of motion are written with the use of the Newton equation for an individual fluid particle. The viscous force $\boldsymbol{f}_{\rm V}$ acting per unit mass of an incompressible fluid is equal to

$$\boldsymbol{f}_{\rm V} = \nu \Delta_{\boldsymbol{X}} \boldsymbol{V} = -\nu \,\mathrm{rot}_{\boldsymbol{X}}\,(\mathrm{rot}_{\boldsymbol{X}}\, \boldsymbol{V}). \tag{1.1}$$

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In view of (1.1), the equations of motion for a fluid particle are written as

$$\boldsymbol{r}_{tt} = -\rho^{-1}\nabla p - \nu \,\mathrm{rot}_{\boldsymbol{X}}\,(\mathrm{rot}_{\boldsymbol{X}}\,\boldsymbol{r}_t), \tag{1.2}$$

where $\boldsymbol{r} = (X, Y, Z)$. We introduce the Jacobi matrix $R = \partial X_i/\partial a_j$ and multiply the left side of Eq. (1.2) by its transposed matrix R^*. As a result, we obtain

$$R^*\boldsymbol{r}_{tt} = -\rho^{-1}\nabla_{\boldsymbol{a}} p - \nu R^* \,\mathrm{rot}_{\boldsymbol{X}}\,(\mathrm{rot}_{\boldsymbol{X}}\,\boldsymbol{r}_t). \tag{1.3}$$

The subscripts $\boldsymbol{a}$ and $\boldsymbol{X}$ at the sign of the vector differential operation indicate that the differentiation is performed over Lagrangian or Eulerian variables, respectively.

To eliminate pressure, we apply a curl operation in the Lagrangian variables to Eq. (1.3):

$$\mathrm{rot}_{\boldsymbol{a}}\,(R^*\boldsymbol{r}_{tt}) = -\nu \,\mathrm{rot}_{\boldsymbol{a}}\,(R^* \,\mathrm{rot}_{\boldsymbol{X}}\,(\mathrm{rot}_{\boldsymbol{X}}\,\boldsymbol{r}_t)). \tag{1.4}$$

Setting $\nu = 0$, using the iquality [5, p. 12]

$$R^*\boldsymbol{r}_{tt} = \frac{\partial}{\partial t}\,(R^*\boldsymbol{r}_t) - \frac{1}{2}\nabla_{\boldsymbol{a}}(|r_t|^2),$$

and integrating over time, we obtain

$$\mathrm{rot}_{\boldsymbol{a}}\,(R^*\boldsymbol{r}_t) = \boldsymbol{S}(a, b, c) \tag{1.5}$$

or, in coordinate form,

$$\begin{gathered} X_{tb}X_c - X_{tc}X_b + Y_{tb}Y_c - Y_{tc}Y_b + Z_{tb}Z_c - Z_{tc}Z_b = S_1(a, b, c), \\ X_{tc}X_a - X_{ta}X_c + Y_{tc}Y_a - Y_{ta}Y_c + Z_{tc}Z_a - Z_{ta}Z_c = S_2(a, b, c), \\ X_{ta}X_b - X_{tb}X_a + Y_{ta}Y_b - Y_{tb}Y_a + Z_{ta}Z_b - Z_{tb}Z_a = S_3(a, b, c). \end{gathered} \tag{1.6}$$

Here S_1, S_2, and S_3 are arbitrary functions of the Lagrangian variables (integrals of motion). Equations (1.6) were first formulated by Cauchy [6]; therefore, the functions S_1, S_2, and S_3 will be referred to as the Cauchy invariants. If the initial positions of the fluid particles coincide with the Lagrangian coordinates, the Cauchy invariants are equal to the initial value of the corresponding vorticity vector components $\boldsymbol{\Omega}$:

$$\Omega_{X0} = S_1, \qquad \Omega_{Y0} = S_2, \qquad \Omega_{Z0} = S_3. \tag{1.7}$$

Generally, in the choice of Lagrangian variables, such a simple relationship between the vorticity vector components and the Cauchy invariants is absent. The quantity S_1 has the meaning of the vorticity flux through unit area in the plane of the Lagrangian variables b and c. The time independence of the function S_1 is due to the conservation of circulation over the infinitesimal contour $\delta b\delta c$ [6]. Similarly, the time independence of S_2 and S_3 is due to the conservation of circulation over the contours $\delta a\delta c$ and $\delta a\delta b$, respectively.

We will treat expression (1.5) as the definition of the Cauchy invariant vector. In the case of a viscous fluid, the quantities S_1, S_2, and S_3 depend on time, but we shall still call them the Cauchy invariants, as in the case of an ideal fluid.

From the calculations performed above, it follows that the left side of Eq. (1.4) is equal to the time derivative of the Cauchy invariant vector:

$$\boldsymbol{S}_t = \mathrm{rot}_{\boldsymbol{a}}\,(R^*\boldsymbol{r}_{tt}). \tag{1.8}$$

Therefore, Eq. (1.4) can be written as

$$\boldsymbol{S}_t = -\nu \,\mathrm{rot}_{\boldsymbol{a}}\,(R^* \,\mathrm{rot}_{\boldsymbol{X}}\,(\mathrm{rot}_{\boldsymbol{X}}\,\boldsymbol{r}_t)). \tag{1.9}$$

The right side of Eqs. (1.9) should be differentiated in the Lagrangian coordinates to take into account viscosity within the framework of the given approach. We use the iquality which is the curl operation in the transformation from one variables to the other [7]:

$$\mathrm{rot}_{\boldsymbol{X}}\,\boldsymbol{A} = \frac{R}{D_0}\,\mathrm{rot}_{\boldsymbol{a}}(R^*\boldsymbol{A}). \tag{1.10}$$

Here $\boldsymbol{A}$ is an arbitrary vector and $D_0 = \det R$ is the Jacobian of the transformation. In the particular case where $\boldsymbol{A} = \boldsymbol{r}_t$, Eq. (1.10) implies the formula

$$\boldsymbol{\Omega} = \frac{R}{D_0}\,\boldsymbol{S}.$$

If the initial positions of the fluid particles coincide with the Lagrangian coordinates, this formula, apparently, becomes relation (1.7).

To write Eq. (1.9) in Lagrangian form, we use the identity (1.10) twice. As a result, we obtain the equation

$$\operatorname{rot}_{\boldsymbol{a}}(R^*\boldsymbol{r}_{tt}) = -\nu \operatorname{rot}_{\boldsymbol{a}}(D_0^{-1}R^*\hat{R}\operatorname{rot}_{\boldsymbol{a}}(D_0^{-1}R^*R\operatorname{rot}_{\boldsymbol{a}}(R\boldsymbol{r}_t))).$$

Taking into account relations (1.7) and (1.8) and introducing the new matrix

$$g = R^*R, \qquad g = g_{ij}, \qquad i,j = 1,2,3,$$

$$g_{11} = X_a^2 + Y_a^2 + Z_a^2, \qquad g_{12} = g_{21} = X_aX_b + Y_aY_b + Z_aZ_b,$$

$$g_{22} = X_b^2 + Y_b^2 + Z_b^2, \qquad g_{13} = g_{31} = X_aX_c + Y_aY_c + Z_aZ_c,$$

$$g_{33} = X_c^2 + Y_c^2 + Z_c^2, \qquad g_{23} = g_{32} = X_bX_c + Y_bY_c + Z_bZ_c,$$

we obtain the equation for the Cauchy invariants:

$$\boldsymbol{S}_t = -\nu \operatorname{rot}_{\boldsymbol{a}}(D_0^{-1}g\operatorname{rot}_{\boldsymbol{a}}(D_0^{-1}g\boldsymbol{S})). \tag{1.11}$$

The determinant D_0 coincides with the right side of the continuity equation. If the initial positions of the particles are taken to be the Lagrangian coordinates, the determinant is equal to unity and the form of Eq. (1.11) is simplified:

$$\boldsymbol{S}_t = -\nu \operatorname{rot}_{\boldsymbol{a}}(g\operatorname{rot}_{\boldsymbol{a}}(g\boldsymbol{S})). \tag{1.12}$$

Equations (1.5) and (1.12) are a new form of the hydrodynamic equations for viscous fluid flows. It should be noted that the differential equation (1.5) in this system of equations is the same as for an ideal fluid. The transformations resulted in the separation of the time scale into inertia scales, which are described by Eq. (1.5), and viscous scales, which are described by Eq. (1.12).

In the case of two-dimensional flows, Eq. (1.12) is slightly simplified. The Cauchy invariant vector has only one component, which is directed, for example, along the c axis (in this case, $\boldsymbol{S} = \boldsymbol{S}_3$). In addition, we take into account that

$$g_{ij}\boldsymbol{S} = \boldsymbol{S}_3,$$

so that Eq. (1.12) ultimately becomes (subscript 3 is omitted):

$$S_t\boldsymbol{c}^0 = -\nu \operatorname{rot}_{\boldsymbol{a}}\Big(D_0^{-1}g\Big(\boldsymbol{a}^0\,\frac{\partial}{\partial b}\,D_0^{-1}S - \boldsymbol{b}^0\,\frac{\partial}{\partial a}\,D_0^{-1}S\Big)\Big). \tag{1.13}$$

Here $\boldsymbol{a}^0$, $\boldsymbol{b}^0$, and $\boldsymbol{c}^0$ are unit vectors of the Lagrangian coordinate system. In scalar form, Eq. (1.13) is written as

$$\begin{gathered} S_t = \{(D_0^{-1}g_{11}(D_0^{-1}S)_b - D_0^{-1}g_{12}(D_0^{-1}S)_a)_b - (D_0^{-1}g_{21}(D_0^{-1}S)_b - D_0^{-1}g_{22}(D_0^{-1}S)_a)_a\}, \\ g_{11} = X_a^2 + Y_a^2, \qquad g_{12} = g_{21} = X_aX_b + Y_aY_b, \qquad g_{22} = X_b^2 + Y_b^2, \\ D_0 = X_aY_b - X_bY_a. \end{gathered} \tag{1.14}$$

Here the subscripts denote differentiation with respect to the corresponding variables. We note that by virtue of the continuity equation, the quantity D_0 does not depend on time. In addition, formulas (1.13) and (1.14) are simplified by setting $D_0 = 1$.

Equations (1.12) and (1.14) are equivalent to system (1), (2) but they do not contain pressure. It should be calculated separately by substituting the solution of Eqs. (1.12) or (1.14) into system (1), (2).

2. Examples of Flows. Both forms of the viscous equation are rather unusual; therefore, it is reasonable to test them for some particular examples. For this, we use the well-known exact solutions that admit explicit analytic representations in the Lagrangian variables. Two of them correspond to the straight-line trajectories of fluid particles, and two to circular trajectories.

2.1. *Flows with Straight-Line Trajectories of Fluid Particles.* Let a flat wall at rest begin to move suddenly at constant velocity U_0 (the first Stokes problem) in its plane. We determine the flow caused by this motion.

We assume that the wall coincides with the xz plane and moves along the x axis. The solution is sought in the form

$$X = a + f(b,t), \qquad Y = b, \quad Z = c. \tag{2.1}$$

This solution satisfies the continuity equation for an arbitrary function f. However, the equation for the Cauchy invariants imposes constraints on the choice of this function. The expressions for R, $\boldsymbol{S}$, and g are written as

$$R = \begin{pmatrix} 1 & f_b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{S} = \begin{pmatrix} 0 \\ 0 \\ -f_{tb} \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & f_b & 0 \\ f_b & 1+f_b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

Substitution of these relations into Eq. (1.12) yields the required equation for the function f:

$$f_{ttb} = \nu f_{tbbb}.$$

Integration of this equation with respect to the Lagrangian variable yields

$$f_{tt} = \nu f_{tbb} + \lambda(a,t). \tag{2.3}$$

From Eqs. (2), it is easy to find that $\lambda(a,t) = -\rho^{-1}p_a$. We assume that the pressure is constant over the entire space. Then, $\lambda = 0$, and the expression (2.3) has the meaning of the heat-conduction equation for the horizontal velocity $X_t = f_t = u(b,t)$. If the wall begins to move suddenly at velocity U_0 (at $t = 0$), the solution of this equation is given by

$$u = U_0 \operatorname{erfc}\eta, \qquad \eta = b/(2\sqrt{\nu t}\,),$$

$$\operatorname{erfc}\eta = \frac{2}{\sqrt{\pi}} \int\limits_{\eta}^{\infty} \exp(-\eta^2)\,d\eta = 1 - \operatorname{erf}\eta = 1 - \frac{2}{\sqrt{\pi}} \int\limits_{0}^{\eta} \exp(-\eta^2)\,d\eta.$$

The single nonzero Cauchy invariant for the given flow is equal to

$$S_3 = \frac{U_0}{\sqrt{\pi\nu t}} \exp\Big(-\frac{b^2}{4\nu t}\Big)$$

and depends on time.

Let an unbounded flat wall performs straight-line harmonic vibrations in its plane at frequency ω (the second Stokes problem). As in the problem considered above, we assume that the x axis is directed along the wall and that the y axis is perpendicular to the wall. The fluid flow is still described by expressions (2.1), and the fluid pressure is considered to be distributed uniformly ($\lambda = 0$). Since the fluid adheres to the wall, the horizontal velocity $u(b,t)$ on its surface obeys the condition $u(b,t)\Big|_{b=0} = u(0,t) = U_0 \cos\omega t$, which is the boundary condition for Eq. (2.3). For Eq. (2.3) with the given boundary conditions, the following expression holds:

$$u(b,t) = U_0\,\mathrm{e}^{-kb} \cos(\omega t - kb), \qquad k = \sqrt{\omega/(2\nu)}.$$

Thus, the fluid particles near the wall perform vibrational motion with the amplitude $U_0 \exp(-kb)$ decreasing with distance from the wall, and the vibrations of the fluid layer at distance b from the wall lag in phase by $b\sqrt{\omega/(2\nu)}$ behind the vibrations of the wall.

For the given type of flow, the Cauchy invariant is equal to

$$S_3 = kU_0\,\mathrm{e}^{-kb}[\cos(\omega t - kb) - \sin(\omega t - kb)].$$

Equations (1.12) do not include pressure. To calculate it, one needs each time to solve the conventional equations of motion in the Lagrangian variables. In this sense, Eqs. (1.12) are incomplete.

2.2. *Flows with Circular Trajectories of Fluid Particles.* We consider flow between two coaxial cylinders rotating at different but constant angular velocities. Let the radii of the inner and outer cylinders be equal to R_1 and R_2, respectively, and let the angular velocities of their rotation be ω_1 and ω_2, respectively. It is more convenient to consider this fluid flow in polar coordinates. We denote the Euler polar coordinates by R_* and Φ and the Lagrangian coordinates by r and φ, so that

$$X = R_* \cos\Phi, \qquad Y = R_* \sin\Phi, \qquad a = r\cos\varphi, \qquad b = r\sin\varphi.$$

We consider flows with circular streamlines where

$$R_* = r, \qquad \Phi = \varphi + f(r,t), \qquad p = p(r,t). \tag{2.4}$$

Writing system (1), (2) in the variables $R_*(r,\varphi)$ and $\Phi(r,\varphi)$ and substituting relations (2.4) into this system, we obtain the following constraints on the choice of f:

$$\rho r f_t^2 = \frac{dp}{dr},$$
$$r f_{tt} = \nu(r f_{trr} + 3 f_{tr}). \tag{2.5}$$

Since f_t is the angular velocity of rotation of the fluid particles, the boundary conditions for this system are given by

$$f_t = \omega_1 \quad \text{at} \quad r = R_1, \qquad f_t = \omega_2 \quad \text{at} \quad r = R_2. \tag{2.6}$$

We assume that f is a linear function of time. Then, the expression for this function subject to constraints (2.6) has the form

$$f_t = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^2 - R_1^2} + \frac{(\omega_1 - \omega_2) R_1^2 R_2^2}{(R_2^2 - R_1^2) r^2}.$$

The Cauchy invariant for this flow is equal

$$S_3 = \frac{1}{r}\frac{\partial}{\partial r}(r^2 f_t) = \frac{2(\omega_2 R_2^2 - \omega_1 R_1^2)}{R_2^2 - R_1^2}$$

and does not depend on time because the flow is steady-state. The same result can be obtained in a simpler way by calculating the vorticity.

Substituting the expression

$$f_t = \frac{\Gamma_0}{2\pi r^2}\Big[1 - \exp\Big(-\frac{r^2}{4\nu t}\Big)\Big] \tag{2.7}$$

in Eq. (2.5), it is easy to see that it is an exact solution of this equation that descries the diffusion of the vorticity from the vortex line with the initial circulation Γ_0 [8].

For the flow (2.7), the Cauchy invariant calculated by formula (1.5) is equal to

$$S_3 = \frac{1}{r}\frac{\partial}{\partial r}(r^2 f_t) = \frac{\Gamma_0}{4\pi\nu t}\exp\Big(-\frac{r^2}{4\nu t}\Big).$$

At the initial time ($t = 0$), the quantity S_3 has a singularity (the vorticity is equal to infinity). Subsequently, the point vortex spreads more and more. The vorticity at the initial location of the vortex becomes finite but it is still maximal in the flow region at the given time.

The invariant S_3 increases in the case of convergent flow and decreases in the case of divergent flow. In addition, the invariant S_3 coincides with the value of the vorticity at the point where the given fluid particle is located.

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